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Where Does Hawking Radiation of a Dynamical Black Hole Come from?

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Abstract Using the gravitational anomaly method proposed by Robinson and Wilczek, Hawking radiation from the apparent horizon of a Vaidya black hole is calculated. The thermodynamics can be built successfully on the apparent horizon. In the meantime, when a time-dependent perturbation is given to the apparent horizon, the first law of thermodynamics can also be constructed successfully at a new supersurface near the apparent horizon. The expressions of the characteristic position and temperature are consistent with the previous result where the viewpoint is that the thermodynamics should be built on the event horizon. Based on the results, the thermodynamics should be constructed on the apparent horizon exactly while the event horizon thermodynamics is just one of the perturbations near the apparent horizon.

Keywords Hawking radiation · Thermodynamics · Dynamical black hole · Apparent horizon · Event horizon

1 Introduction

Hawking's profound discovery of black hole thermal radiation [1, 2] ushered a new era to generate and test ideas concerning the black hole thermodynamics [3, 4] where the surface gravity is regarded as temperature and the area of event horizon is regarded as entropy. As is well known, based on Hawking radiation, the thermodynamical relation has been successfully constructed on relation of the surface gravity κ, the effective area A, and the black hole mass M at the event horizon of the static black holes or stationary black holes while the apparent horizon and event horizon are coincident with each other. However, it will be more

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complex in a dynamical black hole while the event horizon will separate from the apparent horizon. Where does the Hawking radiation come from? Where can the thermodynamics be constructed successfully and perfectly? There are two contrary viewpoints in the previous works. Roberto Balbinot [5], Zhao Zheng [6], and Vagenas et al. [7, 8], have investigated black hole entropy and Hawking radiation at the event horizon of a dynamical black hole. They showed that the radiation from the event horizon will have a corrected temperature expression which is not same as that of a static or stationary black hole. Some of them even think that the radiation is not a perfect black body spectrum, because the thermal equilibrium does not exist on the event horizon. Based on a unique foliation by ingoing null hypersurfaces, Gyula Fodor et al. [9] gave a general expression of surface gravity in a dynamical black hole. When this expression is used to the calculation at the event horizon, the thermodynamics can be built there. On the other hand, Hajicek [10] suggested that the Hawking radiation and thermodynamics should be associated with the apparent horizon instead of the event horizon, since the apparent horizon acts as the boundary of negative energy states. Considering the collapse of a spherical shell, Hiscock [11] proposed to identify one-quarter of the area of the apparent horizon as the Bekenstein-Hawking area-entropy of a Vaidya black hole. Subsequently using a null tetrad formalism, Collins [12] has derived a formula for the area change of the apparent horizon, which can be interpreted as a generalized first law of thermodynamics.

Recently a series of trapped horizon definitions which can be applied to non-equilibrium black holes have been reviewed in [13, 14]. Hayward [15–17], Ashtekar and Krishman [18], and Nielsen [19] have investigated relation of surface gravity, black hole entropy and the total black hole mass, and the thermodynamics can be built on these trapped horizons. Apparent horizon named as marginally outer trapped surface can be seen as one of the trapped horizons, thus these conclusions should be also applicable to the apparent horizon of a dynamical black hole.

How to understand the black hole entropy and Hawking radiation? S. Carlip [20, 21] showed that the breakdown of the diffeomorphism symmetry named as anomalies in quantum field [22, 23] at the horizon played an important role to the Bekenstein-Hawking area-entropy of the horizon. Since the Hawking radiation as well as the black hole entropy is inherent in the horizon, it is natural to expect that the Hawking radiation is also associated with anomalies. Recently in the $(1+1)$-dimensional Schwarzschild spacetime, Robinson and Wilczek [24] proposed a new method to investigate Hawking radiation which can be seen as a cancellation of gravitational anomalies at the horizon. The advantage of these derivations is that it only requires the information at the horizon. So these methods can be applicable to other complex cases including dynamical black holes.

Motivated by the understanding of a dynamical black hole, we will apply the Robinson-Wilczek method to calculate the Hawking radiation from the apparent horizon of a Vaidya black hole. We have seen that Hawking radiation, black hole entropy can be perfectly defined, and the first thermodynamics law can be constructed successfully at the apparent horizon just as Refs. [15–18]. However, considering a time-dependent perturbation near the apparent horizon, we can also construct successfully the first thermodynamics law and the Hawking radiation is very similar as the conclusions in Refs. [5, 6].

In Sect. 2, we will give a brief overview about a Vaidya black hole. The three characteristic supersurfaces will be calculated using the strict definitions. In Sect. 3, using Robinson-Wilczek method we will calculate the Hawking radiation from the apparent horizon of a Vaidya black hole. In Sect. 4, considering a perturbation near the apparent horizon, we will investigate the Hawking radiation from the supersurfaces near the apparent horizon. In Sect. 5, some discussions and conclusions will be given. We will use units in which $c = G = \hbar = k_B = 1$ throughout this paper.

2 The Vaidya Black Hole

The line element of a Vaidya black hole is as follows

$$ds^2 = -\left(1 - \frac{2m(v)}{r}\right)dv^2 + 2dvdr + r^2 d\Omega^2, \tag{1}$$

where $m(v)$ is the mass of the Vaidya black hole and v is Eddington-Finkelstein advanced time coordinate. So $4\pi r^2$ gives the effective area A of a sphere where r is a constant.

The spacetime geometry of a non-stationary black hole usually is characterized by three supersurfaces of particular interest: event horizon, apparent horizon, and time-like limit surface. Thinking of a Vaidya black hole, we can calculate these characteristic supersurfaces as follows.

(1) The event horizon r_{EH}.

It should satisfy the null-surface condition

$$g^{\mu\nu}\frac{\partial f(r,v)}{\partial x^\mu}\frac{\partial f(r,v)}{\partial x^\nu} = 0. \tag{2}$$

After transforming event horizon equation $f(r,v)|_{r_{EH}} = 0$ into the form $r_{EH} = r_{EH}(v)$, then we can rewrite (2) as

$$r_{EH} = \frac{2m(v)}{1 - 2\dot{r}_{EH}}. \tag{3}$$

The surface gravity at the event horizon is given by [9]

$$\kappa|_{r_{EH}} = \frac{m(v)}{r_{EH}^2} = \frac{(1 - 2\dot{r}_{EH})^2}{4m(v)}. \tag{4}$$

(2) The apparent horizon r_{AH}.

It is also called as the outermost marginally trapped surface and it can be defined by the expansion

$$\Theta = l^\mu_{;\mu} - \kappa = 0, \tag{5}$$

where $\kappa = n^\mu l^\nu l_{\mu;\nu} = n^\mu l^\nu (l_{\mu,\nu} - \Gamma^\sigma_{\mu\nu} l_\sigma)$ can be regarded as surface gravity of the horizon.

In order to calculate the apparent horizon, we need to choose a null tetrad frame. We can choose it as

$$n_\mu = (1,0,0,0), \quad l_\mu = \left(\frac{1}{2}\left(1 - \frac{2m(v)}{r}\right), -1, 0, 0\right), \quad m_\mu = \sqrt{\frac{1}{2}}(0,0,r,ir\sin\theta),$$
$$\overline{m}_\mu = \sqrt{\frac{1}{2}}(0,0,r,-ir\sin\theta). \tag{6}$$

So the contravariant forms of the basis vectors are

$$n^\mu = (0,-1,0,0), \quad l^\mu = \left(1, \frac{1}{2}\left(1 - \frac{2m(v)}{r}\right), 0, 0\right),$$
$$m^\mu = \sqrt{\frac{1}{2}} r^{-1}\left(0,0,1,\frac{i}{\sin\theta}\right), \quad \overline{m}^\mu = \sqrt{\frac{1}{2}} r^{-1}\left(0,0,1,-\frac{i}{\sin\theta}\right). \tag{7}$$

It is easy to see that they satisfy the condition of the null tetrad frame

$$n_\mu n^\mu = l_\mu l^\mu = m_\mu m^\mu = \overline{m}_\mu \overline{m}^\mu = 0,$$
$$n_\mu l^\mu = -m_\mu m^\mu = 1,$$
$$n_\mu m^\mu = n_\mu \overline{m}^\mu = l_\mu m^\mu = l_\mu \overline{m}^\mu = 0.$$

Substituting (6) and (7) into (5), we can have

$$r_{AH} = 2m(v), \tag{8}$$

then we have the surface gravity at the apparent horizon as [9]

$$\kappa = \frac{m(v)}{r^2}\bigg|_{r_{AH}} = \frac{1}{4m(v)}. \tag{9}$$

(3) The timelike limit surface r_{TLS}.
It satisfies

$$g_{vv} = -\left(1 - \frac{2m(v)}{r_{TLS}}\right) = 0,$$

so we have

$$r_{TLS} = 2m(v). \tag{10}$$

Apparently for a Vaidya black hole, we have the relation $r_{AH} = r_{TLS}$.

3 Hawking Radiation from the Apparent Horizon

Considering the metric of a Vaidya black hole, we can see that the scalar field theory on this metric can be reduced to the 2-dimensional theory. The action of the scalar field in a Vaidya spacetime is

$$S[\phi, g_{\mu\nu}] = \frac{1}{2}\int d^4x\sqrt{-g}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$$
$$= \frac{1}{2}\int dvdrd\theta d\varphi r^2\sin\theta\phi\left[2\partial_r\partial_v + \left(1 - \frac{2m(v)}{r}\right)\partial_r^2 + \frac{1}{r^2}\triangle\Omega\right]\phi.$$

Thinking of the region near the apparent horizon $r_{AH} = 2m(v)$, we will find that only dominant terms are left. Thus the action becomes

$$S[\phi] = \frac{r_{AH}^2}{2}\int dvdrd\theta d\varphi\sin\theta\phi\left(2\partial_r\partial_v + \left(1 - \frac{2m(v)}{r}\right)\partial_r^2\right)\phi$$
$$= \sum_{lm}\frac{r_{AH}^2}{2}\int dvdr\phi_{lm}\left(2\partial_r\partial_v + \left(1 - \frac{2m(v)}{r}\right)\partial_r^2\right)\phi_{lm}. \tag{11}$$

In the second line, $\phi = \sum_{l,m}\phi_{lm}Y_{lm}$ has been expanded by spherical harmonics. It is straightforward to show that the action can be effectively described by an infinite collection of the scalar fields on the 2-dimensional metric

$$ds^2 = -\left(1 - \frac{2m(v)}{r}\right)dv^2 + 2dvdr. \tag{12}$$

Thus, we can reduce the scalar field theory in 4-dimensional black hole spacetime to that in 2-dimensional spacetime near the apparent horizon. In this 2-dimensional spacetime, we treat the black hole horizon as the boundary of the spacetime and discard ingoing modes near the horizon that cannot effect the dynamics of the scalar fields out of the horizon. The 2-dimensional theory is chiral. Subsequently, we split the region into two: $r_{AH} \le r \le r_{AH}+\epsilon$, where the theory is chiral and $r \ge r_{AH}+\epsilon$, where the theory is not chiral. We will take the limit $\epsilon \to 0$ ultimately.

The gravitational anomaly in 2-dimensional chiral theory takes the form as

$$\nabla_\mu T^\mu_\nu = \frac{1}{96\pi\sqrt{-g}} \epsilon^{\beta\delta} \partial_\delta \partial_\alpha \Gamma^\alpha_{\nu\beta}. \tag{13}$$

We define A_μ and N^μ_ν as

$$\nabla_\mu T^\mu_\nu = A_\nu = \frac{1}{\sqrt{-g}} \partial_\mu N^\mu_\nu . \tag{14}$$

In the region of $r \ge r_{AH}+\epsilon$, A_ν =0, $N^\mu_\nu = 0$. However, near the apparent horizon, $r_{AH} \le r \le r_{AH}+\epsilon$, we can get

$$\begin{aligned} N^r_v &= \frac{1}{96\pi} \epsilon^{\beta r} \partial_\alpha \Gamma^\alpha_{v\beta} = \frac{1}{96\pi} \epsilon^{vr} \partial_\alpha \Gamma^\alpha_{vv} \\ &= \frac{1}{96\pi} \left(\frac{6m^2(v)}{r^4} - \frac{2m(v)}{r^3} \right). \end{aligned} \tag{15}$$

The effective action for the metric $g_{\mu\nu}$ after integrating out the scalar field is

$$W[g_{\mu\nu}] = -i \ln\left(\int D\phi \cdot e^{iS[\phi, g_{\mu\nu}]} \right), \tag{16}$$

where $S[\phi, g_{\mu\nu}]$ is the classical action. After the infinitesimal general coordinate transformation $x^\mu \to x^\mu - \lambda^\mu$, the variation of the effective action becomes

$$\begin{aligned} -\delta_\lambda W &= -\delta_\lambda S = -\int d^2x \sqrt{-g} \lambda^\nu \nabla_\mu T^\mu_\nu \\ &= \int d^2x \sqrt{-g} \lambda^\nu \nabla_\mu \{ T^\mu_{(H)\nu} H(r) + T^\mu_{(O)\nu} \Theta_+(r) \} \\ &= \int d^2x \sqrt{-g} \lambda^\nu \left\{ \partial_r (N^r_\nu H(r)) + \left(T^r_{(O)\nu} - T^r_{(H)\nu} + \frac{1}{\sqrt{-g}} N^r_\nu \right) \times \delta(r - r_{AH} - \epsilon) \right\} \\ &\quad + \int d^2x \sqrt{-g} \lambda^r (T^r_{(O)r} - T^r_{(H)r}) \times \delta(r - r_{AH} - \epsilon), \end{aligned} \tag{17}$$

where $\Theta_+(r) = \Theta(r - r_{AH} - \epsilon)$, and $H(r) = 1 - \Theta_+(r)$. The subscript H and O denote the value in the region $r_{AH} \le r \le r_{AH}+\epsilon$ and $r \ge r_{AH}+\epsilon$ respectively. We have also considered the fact that $T^\mu_{(O)\nu}$ is covariantly conserved and $T^\mu_{(H)\nu}$ is anomalous in (13).

Taking into account the time variation independence of $T^\mu_{(O)\nu}$ and $T^\mu_{(H)\nu}$ in a certain time v, we can integrate (14). In the limit $r \to r_{AH}$, we have

$$T^r_{(H)v} = -K_{(H)}, \quad T^r_{(O)v} = -K_{(O)},$$

$$T^r_{(H)r} = \frac{K_{(H)} + Q_{(H)}}{1 - \frac{2m(v)}{r}}, \quad T^r_{(O)r} = \frac{K_{(O)} + Q_{(O)}}{1 - \frac{2m(v)}{r}}. \tag{18}$$

Substituting (18) into (17) and taking the limit $\epsilon \to 0$, we obtain

$$\delta_\lambda W = \int d^2x\sqrt{-g}\lambda^v \left\{\partial_r(N^r_v H) + \left(-K_{(H)} + K_{(O)} + \frac{1}{\sqrt{-g}}N^r_v\right) \times \delta(r - r_{AH})\right\}$$
$$+ \int d^2x\sqrt{-g}\lambda^r \frac{K_{(H)} + Q_{(H)} - K_{(O)} - Q_{(O)}}{1 - \frac{2m(v)}{r}}\delta(r - r_{AH}). \tag{19}$$

Considering the first term should be canceled by the quantum effect of the ingoing modes, it should be ignored. When $\delta_\lambda W = 0$, we get

$$K_{(O)} = K_{(H)} - \Phi, \qquad Q_{(O)} = Q_{(H)} + \Phi, \tag{20}$$

where

$$\Phi = \frac{1}{\sqrt{-g}}N^r_v \,|_{r_{AH}} = \frac{1}{96\pi}\left(\frac{6m^2(v)}{r^4} - \frac{2m(v)}{r^3}\right)|_{r_{AH}} = \frac{1}{768\pi m^2(v)}. \tag{21}$$

Considering $\kappa = \frac{1}{4m(v)}$, it implies the following Hawking temperature

$$T_{AH} = \frac{\kappa}{2\pi} = \frac{1}{8\pi m(v)}. \tag{22}$$

On the other hand, a beam of massless black body radiation moving outwards in the radial direction at a temperature T_H has a flux of the form

$$\Phi = \frac{\pi}{12}T^2_{AH}, \tag{23}$$

it is evident that the flux as (21) is nothing but the Hawking flux, which cancels the gravitational anomaly.

According to the thermodynamical analogy in black hole physics, the entropy of the black hole is defined as $S = \frac{A}{4}$, where A is the area of black hole characteristic supersurface. Now we assume that it is also available to the apparent horizons [10]. Thus the entropy can be written as

$$S_{BH} = \pi r^2_{AH} = 4\pi m^2(v), \tag{24}$$

which satisfies the first law of the thermodynamics

$$dm(v) = \frac{\kappa}{2\pi}dS_{BH} = T_{AH}dS_{BH}, \tag{25}$$

where κ is just the surface gravity of the apparent horizon from the paper [9] as (9).

4 Hawking Radiation from the Supersurfaces near the Apparent Horizon

Subsequently, we investigate Hawking radiation from the supersurfaces near the apparent horizon, where we assume $r' = r_{AH}(1 + \delta)$, and δ is an infinitesimal parameter. Firstly, we define a coordinate transformation as

$$R = r - \frac{1}{2}\delta \cdot v, \tag{26}$$

so the line element can be written as

$$ds^2 = -\left(1 - \frac{2m(v)}{r} - \delta\right)dv^2 + 2dvdR + r^2 d\Omega^2. \tag{27}$$

It is obviously that $r = r' = 2m(v)(1+\delta) = r_{AH}(1+\delta)$ can be regarded as a new "apparent horizon", which satisfies

$$1 - \frac{2m(v)}{r'} - \delta = 0.$$

Using Robinson-Wilczek method, after similar calculation as from (13) to (21), we can obtain Hawking radiation flux from $r' = r_{AH}(1+\delta)$ as

$$\Phi' = \frac{1}{\sqrt{-g}} N_v^R \,|_{r'} = \frac{1}{768\pi m^2(v)(1+\delta)^2}. \tag{28}$$

In the meantime, we have the surface gravity and thermodynamical temperature expressions on the new supersurface as

$$\kappa' = \frac{1}{4m(v)(1+\delta)^2}, \qquad T' = \frac{1}{8\pi m(v)(1+\delta)^2}. \tag{29}$$

Thinking of (23), we have got the Hawking radiation as (28) on the supersurface $r' = r_{AH}(1+\delta)$ which cancels the gravitational anomaly.

If we set $\delta = 2\dot{r}_{EH} \approx 4\dot{m}(v)$, we can get

$$T' = T_{EH} = \frac{(1-2\dot{r}_{EH})^2}{8\pi m(v)} \approx \frac{1}{8\pi m(v)}(1 - 8\dot{m}),$$

which is just the Hawking temperature from event horizon obtained by the papers [5, 6] and the expression of (29) will be exactly (4) as the surface gravity and temperature on the event horizon.

The area of the supersurface $r' = r_{AH}(1+\delta)$ that can be regarded as the entropy of the black hole thermal system will naturally be

$$S' = \frac{1}{4}A' = \pi r'^2 = \pi(2m(v)(1+\delta))^2. \tag{30}$$

From (29) and (30), they obey the first law of thermodynamics as

$$dm(v) = T' dS' = \frac{1}{8\pi m(v)(1+\delta)^2} \cdot 8\pi m(v)(1+\delta)^2 dm(v).$$

Apparently, we can also construct thermodynamics on the new supersurface which is a perturbation near the apparent horizon.

5 Discussions and Conclusions

Although a perfect quantum gravitational theory has not been constructed, some quantum effects as Hawking radiation in a black hole have been investigated successfully using some methods in the curved spacetime quantum field theory. Due to a non-stationary Vaidya black hole, some properties can be seen as follows.

Firstly, from (25) it is clear to see that the first law of thermodynamics can be constructed on the apparent horizon, and this is consistent with the conclusions from previous works [15–18]. The apparent horizon named as marginally outer trapped surface can be regarded as one case of the trapped horizons.

Secondly, we find that the first law of thermodynamics can also apply to the event horizon. Moreover, considering a time-dependent perturbation near the apparent horizon, we can also construct successfully black hole thermodynamics on the new supersurface. That is to say, the event horizon is just one of the many new supersurfaces via time-dependent perturbation around the apparent horizon.

Thirdly, due to our derivations of Hawking radiation, it is obviously that our conclusions should be applicable to other non-stationary black holes, at least for the spherical dynamical black holes.

Based on the results above, the thermodynamics should be constructed on the apparent horizon exactly, while the event horizon thermodynamics is just one of the perturbations near the apparent horizon.

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References

1. Hawking, S.W.: Nature **248**, 30 (1974)
2. Hawking, S.W.: Commun. Math. Phys. **43**(3), 199 (1975)
3. Bekenstein, J.D.: Phys. Rev. D **7**, 2333 (1973)
4. Bardeen, J.M., Carter, B., Hawking, S.W.: Commun. Math. Phys. **31**, 161 (1973)
5. Balbinot, R.: Phys. Rev. D **33**, 1611 (1986)
6. Jun, R., Zhang, J.-Y., Zhao, Z.: Chin. Phys. Lett. **23**, 2006 (2019)
7. Vagenas, E.C., Das, S.: J. High Energy Phys. **0610**, 025 (2006)
8. Vagenas, E.C., Das, S.: Gravitational anomalies, Hawking radiation, and spherically symmetric black holes. arxiv: hep-th/0606077 (2006)
9. Fodor, G., Nakamura, K., Oshiro, Y., Tomimatsu, A.: Phys. Rev. D **54**, 3882 (1996)
10. Hajicek, P.: Phys. Rev. D **36**, 1065 (1987)
11. Hiscock, W.A.: Phys. Rev. D **40**, 1336 (1989)
12. Collins, W.: Phys. Rev. D **45**, 495 (1992)
13. Gourgoulhon, E., Jaramillo, J.L.: New Astron. Rev. **51**, 791 (2008)
14. Gourgoulhon, E., Jaramillo, J.L.: New theoretical approaches to black holes. arXiv:0803.2944v2 (2008)
15. Hayward, S.A.: Phys. Rev. D **49**, 6467 (1994)
16. Hayward, S.A.: Phys. Rev. Lett. **93**, 251101 (2004)
17. Hayward, S.A.: Phys. Rev. D **70**, 104027 (2004)
18. Ashtekar, A., Krishman, B.: Phys. Rev. D **68**, 104030 (2003)
19. Nielsen, A.B.: Black holes and black hole thermodynamics without event horizons. arXiv:0809.3850v1 (2008)
20. Carlip, S.: Int. J. Theor. Phys. **46**, 2192 (2007)
21. Carlip, S.: Horizons, constraints, and black hole entropy. arxiv: gr-qc/0601041 (2006)
22. Bertlmann, R.: Anomalies in Quantum Field Theory. Oxford Sciences, Oxford (2000)
23. Fujikawa, K., Suzuki, H.: Path Integrals and Quantum Anomalies. Oxford Sciences, Oxford (2004)
24. Robinson, S.P., Wilczek, F.: Phys. Rev. Lett. **95**, 011303 (2005)